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A Structure Theory of Bipartite Graphs of Finite Exterior Dimension

A. L. DULMAGE AND N. S. MENDELSOHN, F.R.S.C.

1. Introduction and summary. In **(3**, §§4, 5, 6**)**, the authors presented a partial structure theory for bipartite graphs. This paper is a continuation of that work. The following results are achieved. Any bipartite graph of finite exterior dimension is decomposed in a unique way into a union of core-graphs no two of which have an edge in common. The core-graphs are then decomposed into a disjoint union of irreducible and minimal semi-irreducible subgraphs, the decomposition again being unique.

Since a graph is a loose mathematical structure in the sense that new graphs may be formed from it by the addition or removal of edges in an arbitrary manner, a question of interest is how do such changes affect the structure of the graph. This problem is discussed in §5.

Finally, a study is made of irreducible graphs. An irreducible graph is formed in a peculiar way from simple irreducible graphs and a description of the method of formation of such graphs is given.

2. Notation and definitions. Unless otherwise stated all terms used have the same meaning as given in **(3)** which is prerequisite for this paper. It has been necessary or convenient to modify a couple of definitions and conventions and also to introduce a number of new terms.

A bipartite graph is a system consisting of a pair of vertex sets S and T and a set of edges K, each edge of K being a pair (s, t) with $s \in S$ and $t \in T$. The set of all pairs (s, t) with $s \in S$ and $t \in T$ is denoted by $S \times T$. Throughout this paper the term graph will be used instead of bipartite graph and, unless otherwise stated, the graph is to be considered as being the set of its edges. If S and T are finite, $S = (a_1, a_2, \ldots, a_m)$, $T = (b_1, b_2, \ldots, b_n)$, it is convenient to represent K as a set of places in an $n \times m$ matrix $A = (a_{rs})$. It is usual to use an actual matrix to represent K by putting $a_{ij} = 1$ if (a_i, b_j) is an edge of K and $a_{ij} = 0$ if (a_i, b_j) is not an edge of K. More generally K may be represented by a matrix A with $a_{ij} \neq 0$ if $(a_i, b_j) \in K$, otherwise $a_{ij} = 0$. Still more generally, and this includes the case where S and T may be infinite, the edges of K may be represented schematically as points in a Cartesian plane with the set S represented as a vertical axis pointing downward and T as a horizontal axis pointing to the right. (This convention differs from that used in **(3)**, but it is being used

because it makes the matrix and Cartesian representations consistent with each other.) Because of these representations some of the language of matrices will be used. For example, edges (s, t_1) and (s, t_2) are said to be in the same row and edges (s_1, t), (s_2, t) are said to be in the same column. Analogously, when a set of edges is said to be contained in a square or in a rectangular block, what is referred to is the set of places occupied by the edges (if necessary, after rearrangement of rows and columns) in the matrix or Cartesian representation. If two graphs K_1 and K_2 have no vertices or edges in common they are said to be disjoint and the graph K which consists of all the vertices and edges of K_1 and K_2 is said to be their disjoint sum. The notation $K = K_1 \oplus K_2$ indicates that K is the disjoint sum of K_1 and K_2 and the definition is extended in the obvious way to represent a disjoint sum $K_1 \oplus K_2 \oplus K_3 \oplus \ldots \oplus K_n$ of n graphs. If K_1 and K_2 have no edges in common their union K will be written as $K = K_1 + K_2$. After rearrangement of rows and columns a disjoint sum will appear as a graph contained in diagonally opposite blocks in the Cartesian representation. The word disjoint will also be used to indicate that subsets A, B, C, D, . . . , of the vertex sets S or T have no element in common. If A is a subset of S, $\bar{A}$ will denote the complement of A in S and the same notation will represent complementation with respect to T. If A is a subset of S or T, the number of elements in A is denoted by $\nu(A)$ with $\nu(A) = \infty$ if A is an infinite set. A graph K is said to be covered by the pair $[A, B]$ if A and B are subsets of S and T respectively and if, for each edge (s, t) of K, either $s \in A$ or $t \in B$ or both. If K has a cover $[A, B]$ with $\nu(A) + \nu(B)$ finite, K is said to be of finite exterior dimension. This dimension is an integer denoted by $E(K)$ and its value is $\min(\nu(A) + \nu(B))$ taken over all covers $[A, B]$ of K. Any cover $[A, B]$ at which the minimum value of $\nu(A) + \nu(B)$ is attained is called a minimal exterior pair, abbreviated as m.e.p. The exterior dimension of a graph may also be called its term rank, this latter term having been introduced into the literature from a study of the matrix representation. A pair $\{A, B\}$ is said to be interior to K if each edge (a, b) with $a \in A$ and $b \in B$ is an edge of K. The number $\max(\nu(A) + \nu(B))$ taken over all interior pairs $\{A, B\}$ of K is called the interior dimension of K.

In **(3)**, the authors have introduced the concept of an irreducible graph. In this paper it has been found desirable to modify the definition. Denoting the null set of S or T by ϕ, a graph K is said to be *reducible* if it has an m.e.p. $[A, B]$ in which $A \neq \phi$ and $B \neq \phi$. A graph K is said to be *semi-irreducible* if there is a pair $[A, B]$ in which $A = \phi$ or $B = \phi$ which is an m.e.p. for K and if K has no other m.e.p. A graph K is said to be *irreducible* if it has exactly two m.e.p.'s, namely, a pair $[A, B]$ in which $A = \phi$ and a pair $[A, B]$ in which $B = \phi$. Thus a graph K is either reducible, semi-irreducible, or irreducible, but no graph can be two of these.

If K is of finite exterior dimension and irreducible it must be contained in a square block (with no empty rows or columns). If K is a semi-irreducible graph, then the smallest rectangular block which contains K must be non-

square of dimensions n by m, and the exterior dimension of K is equal to the lesser of n and m (one of n and m may be infinite). If K is a semi-irreducible graph of exterior dimension n, it is not necessarily true that K contains a subgraph of exterior dimension n which is irreducible. This is shown in the following example.

Example. Let $S = (a_1, a_2, a_3)$, $T = (b_1, b_2, b_3, b_4, b_5)$ and let the edges of K be (a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_5), (a_3, b_3), (a_3, b_5). Here K has only $[S, \phi]$ as its m.e.p. but every subgraph of exterior dimension 3 is reducible.

Some semi-irreducible graphs are expressible as disjoint sums of smaller semi-irreducible graphs. Any semi-irreducible graph which cannot be so expressed is called a ***minimal semi-irreducible graph.***

A concept which is useful in the study of the structure of irreducible graphs is that of an induced graph. Let K be a graph in the space $S \times T$ and let S and T each be decomposed into a union of disjoint subsets $S = \cup \alpha_i$, $T = \cup \beta_j$. A graph K^* is constructed as follows: the vertex sets S^* and T^* have as their elements the sets α_i and β_j respectively. An edge (α_i, β_j) belongs to K^* if and only if K contains at least one edge (a, b) with $a \in \alpha_i$ and $b \in \beta_j$. The graph K^* is said to be induced from K by the partitions S^* and T^* of S and T. In the matrix representation of a graph the process of induction consists of partitioning of the matrix (after rearrangement of rows and columns) into rectangular blocks by horizontal and vertical lines, a block of the partitioned matrix being an edge of the induced graph if and only if this block contains an edge of the original block. Let $A_1 \times B_1, A_2 \times B_2, \ldots, A_r \times B_r$ be disjoint subregions of $S \times T$ (that is, $A_1, A_2, \ldots, A_r$ are disjoint subsets of S and $B_1, B_2, \ldots, B_r$ are disjoint subsets of T). The region

$$(A_1 \cup A_2 \cup A_3 \ldots \cup A_r) \times (B_1 \cup B_2 \cup B_3 \ldots \cup B_r)$$

is the smallest rectangular region containing $A_1 \times B_1, A_2 \times B_2, \ldots, A_r \times B_r$. The blocks $A_1 \times B_1, A_2 \times B_2, \ldots, A_r \times B_r$ will be called a diagonal of blocks of

$$(A_1 \cup A_2 \ldots \cup A_r) \times (B_1 \cup B_2 \ldots \cup B_r)$$

and each of the regions $A_i \times B_j$ $(i, j = 1, 2, \ldots, r; i \neq j)$ will be said to be adjacent to the diagonal.

Another concept useful in the study of the structure of an irreducible graph is that of a cycle. A cycle in a bipartite graph K is a finite subgraph $K^\#$ with the following properties. Let S and T be the vertex sets of K. If (s_1, t_1) is an edge of $K^\#$ then there exists exactly one vertex $s_2 \in S$, $s_2 \neq s_1$, such that (s_2, t_1) is an edge of $K^\#$, and there exists exactly one vertex $t_2 \in T$, $t_2 \neq t_1$, such that (s_2, t_2) is an edge of $K^\#$, and there exists exactly one vertex $s_3 \in S$, $s_3 \neq s_2$ such that (s_3, t_2) is an edge of $K^\#$, etc. If after $2k-1$ steps, $k \geqslant 2$, it is found that (s_1, t_1), (s_2, t_1), $(s_2, t_2), \ldots, (s_k, t_k)$, (s_1, t_k), are distinct and are all the edges of $K^\#$ then $K^\#$ is a cycle of rank k. The rank

of a cycle is the same as its exterior dimension and it will be shown that a cycle is an irreducible graph. If K is a graph each of whose vertex sets contains exactly n elements and K contains a cycle of rank n, K is said to be a simple irreducible graph. The method by which any irreducible graph is constructed from simple irreducible graphs will be described further on.

3. The basic structure theory. Let K be a graph of finite exterior dimension and let S and T be its vertex sets. By (**3**, p. 523), following Theorem 7, K produces disjoint decompositions of S and T denoted by

$$S = A_* \cup S_1 \cup S_2 \ldots \cup S_k \cup \bar{A}^*; \; T = B_* \cup T_k \cup T_{k-1} \ldots \cup T_1 \cup \bar{B}^*$$

which in turn divide the product space $S \times T$ into three regions

$$R_1 = (A_* \times \bar{B}^*) \cup (S_1 \times T_1) \cup (S_2 \times T_2) \ldots \cup (\bar{A}^* \times B_*)$$

$$R_2 = (A_* \times B^*) \cup (A^* \times B_*) \bigcup_{i<j} (S_i \times T_j)$$

$$R_3 = (\bar{A}_* \times \bar{B}^*) \cup (\bar{A}^* \times \bar{B}_*) \bigcup_{i>j} (S_i \times T_j).$$

The regions R_2 and R_3 are not uniquely determined, since, in the construction of the decomposition of S and T, there is a certain amount of flexibility in the order in which the sets S_i, T_j make their appearance, but this is of no real importance. The subgraph $K \cap R_1$ which will be denoted by C_1 is called the first core of K and has the following properties.

(1) If K is non-null, C_1 is non-null.

(2) All but a finite number of edges of K belong to C_1.

(3) C_1 consists of all the admissible edges (**3**, p. 519) of K.

(4) $E(K) = E(C_1)$.

The partition of S and T decomposes C_1 as a disjoint sum of graphs as follows:

$$C_1 = \{K \cap (A_* \times \bar{B}^*)\} \oplus \{K \cap (S_1 \times T_1)\} \oplus \ldots \oplus \{K \cap (S_k \times T_k\} \oplus \{K \cap (\bar{A}^* \times B_*)\}$$

with the further following properties.

(1) Each of the subgraphs

$$G_i^{(1)} = K \cap (S_i \times T_i), \qquad i = 1, 2, \ldots, k$$

is irreducible.

(2) Each of the graphs $K \cap (A_* \times \bar{B}^*)$ and $K \cap (\bar{A}^* \times B_*)$ is semi-irreducible.

The subgraphs $K \cap (A_* \times \bar{B}^*)$ and $K \cap (\bar{A}^* \times B_*)$ will be called tails and these tails can be further reduced as follows. Since $K \cap (A_* \times \bar{B}^*)$ is semi-irreducible with $\nu(A_*)$ finite and with $[A_*, \phi]$ its only m.e.p., it can be decomposed into a disjoint sum of a finite number m of minimal semi-irreducible subgraphs. This decomposition is described in the following

way. The vertex sets A_* and $\bar{B}^*$ are each decomposed into m mutually disjoint subsets;

$$A_* = U_1 \cup U_2 \cup U_3 \ldots \cup U_m$$
$$\bar{B}^* = V_1 \cup V_2 \cup V_3 \ldots \cup V_m,$$

and

$$K \cap (A_* \times \bar{B}^*) = \{K \cap (U_1 \times V_1)\} \oplus \{K \cap (U_2 \times V_2)\} \ldots \oplus \{K \cap (U_m \times V_m)\}.$$

Each component $H_i^{(1)} = K \cap (U_i \times V_i)$ is a minimal semi-irreducible subgraph with the properties: (1) $\nu(U_i) < \nu(V_i)$ ($\nu(V_i)$ may be infinite), (2) the only m.e.p. for $H_i^{(1)}$ is $[U_i, \phi]$, and (3) $E(H_i^{(1)}) = \nu(U_i)$. Similarly, the other tail $K \cap (\bar{A}^* \times B_*)$ can be expressed as a disjoint union of r minimal semi-irreducible subgraphs

$$L_i^{(1)} = K \cap (X_i \times W_i) \qquad (i = 1, 2, \ldots, r)$$

with the properties (1) $\nu(X_i) < \nu(W_i)$ ($\nu(X_i)$ may be infinite), (2) the only m.e.p. for $L_i^{(1)}$ is $[\phi, W_i]$, and (3) $E(L_i^{(1)}) = \nu(W_i)$. The total decomposition of C_1 is illustrated in Figure 1 (the figure is drawn as though V_1, $V_2, \ldots, V_{m-1}$, $X_1, X_2, \ldots, X_{r-1}$ are finite, though this inference should not be made).

The decomposition of K is now extended as follows. The subgraph $K - C_1 = K \cap R_2$ has the property $E(K - C_1) < E(K)$. Let C_2 be the core of $K - C_1$. C_2 is now decomposed into a disjoint sum of irreducible and minimal semi-irreducible subgraphs in the same way as C_1 was. These subgraphs are denoted by $G_i^{(2)}$, $H_i^{(2)}$ and $L_i^{(2)}$. Also

$$E(C_2) = E(K - C_1) < E(K) = E(C_1).$$

Repeating the process with $K - (C_1 + C_2)$ to obtain C_3 etc. a sequence of subgraphs $C_1, C_2, C_3, \ldots$, is obtained. The subgraph C_i is called the ith core of K. Since

$$E(K) = E(C_1) > E(C_2) > E(C_3) \ldots \geqslant 0$$

the process stops after a finite number of steps. Summarizing these results yields the following theorem:

THEOREM 1. *Any graph K of finite exterior dimension can be decomposed into a finite number of cores $K = C_1 + C_2 + \ldots + C_t$. Each core C_i can be decomposed into a disjoint sum of a finite number of irreducible and minimal semi-irreducible subgraphs*

$$C_i = \sum_j G_j^{(i)} \oplus \sum_p H_p^{(i)} \oplus \sum_q L_q^{(i)}.$$

Furthermore,

$$E(K) = E(C_1) > E(C_2) > E(C_3) \ldots > E(C_t).$$

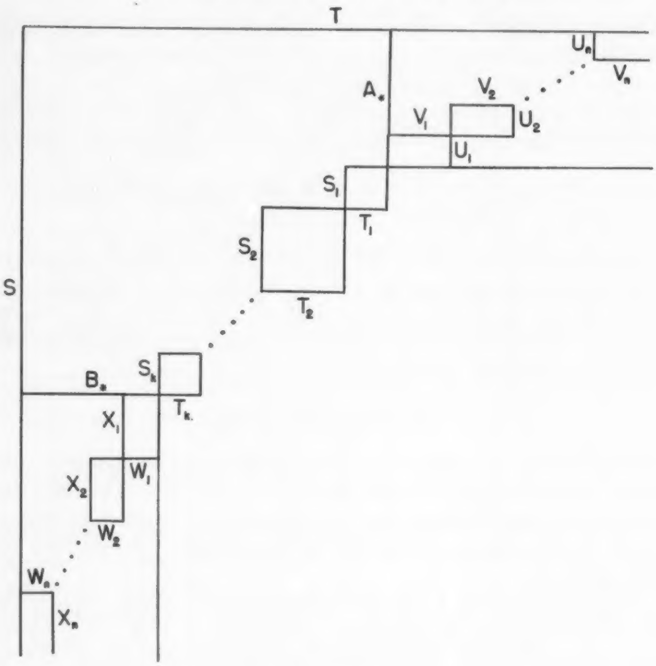

FIGURE 1

By Theorem 1, each edge of K is placed in a unique square or rectangular block (for example, each edge of $H_p^{(1)}$ is in the block $U_p \times V_p$). These blocks will be referred to as the blocks of the graph.

4. The structure of irreducible graphs. An irreducible graph, by our definition is a finite graph whose vertex sets have the same number of elements. Furthermore, the exterior dimension of an irreducible graph is the number of elements in one of the vertex sets. Accordingly, let $S = (a_1, a_2, \ldots, a_n)$ and $T = (t_1, t_2, \ldots, t_n)$ be the vertex sets of a graph K of exterior dimension n. By Theorem 1, of **(3)**, K contains at least one subgraph K^* containing exactly n edges and such that $E(K^*) = n$. In this section, any such subgraph K^* will be called a diagonal. In particular, without loss of generality, we may assume that a diagonal K^* is the subgraph $(s_1, t_1), (s_2, t_2), \ldots, (s_n, t_n)$ of K. A necessary condition that K be irreducible is that every row and every column of $S \times T$ contains at least two edges of K. For if (s, t) is an edge of K and this is the only edge in its row, then the pair $[A - s, t]$ is an m.e.p. for K, so that K is reducible.

THEOREM 2. *If K is a graph with vertex sets S and T such that $\nu(S) = \nu(T) = n$ and if K contains a cycle of rank n, then K is an irreducible graph.*

Proof. $E(K) = n$ since the cycle of K contains a diagonal of length n. If K is reducible, the space $S \times T$ contains a block $A \times B$ such that $\nu(A) + \nu(B) = n$, $\nu(A) \neq 0$, $\nu(B) \neq 0$ and such that $A \times B$ contains no edge of K (**3**, p. 520, Theorem 4). Let K^* be a cycle of rank n contained in K. Each row and each column of $S \times T$ contains two edges of K^*. Hence $(\bar{A} \times B)$ contains $2\,\nu(B)$ edges of K^* and $(A \times \bar{B})$ contains $2\,\nu(A)$ edges of K^*. Thus all the edges of K^* are contained in $(\bar{A} \times B) \cup (A \times \bar{B})$. Hence K^* cannot be a cycle since it contains no edges in either of the non-null blocks $A \times B$ or $\bar{A} \times \bar{B}$ a contradiction.

An alternative proof of Theorem 2 can be given along the following lines. To show that K is irreducible it is sufficient to show that any pair (s, t) of $S \times T$ is admissible, that is, if (s, t) were added to K (if necessary) it would be an admissible edge. If $(s, t) \in K^*$ it belongs to a diagonal of K. If $(s, t) \bar{\in} K^*$ remove from K^* the edges (s, t_1), (s, t_2), (s_1, t), (s_2, t). A diagonal of K containing (s, t) is formed as follows. Include in the diagonal the edge (s, t) together with those edges of K^* which are in the same row or column as a removed edge. The remainder of the diagonal can always be formed with edges in K^*. The theorem follows by Theorem 3 of (**3**).

Theorem 2 gives a condition for irreducibility which is sufficient but not necessary as the following example shows:

Example. Let $S = (a_1, a_2, \ldots, a_{10})$, $T = (b_1, b_2, \ldots, b_{10})$, $K = (a_1, b_1)$, (a_1, b_2), (a_2, b_2), (a_2, b_3), (a_2, b_7), (a_3, b_3), (a_3, b_4), (a_4, b_4), (a_4, b_5), (a_5, b_1), (a_5, b_5), (a_6, b_6), (a_6, b_7), (a_7, b_7), (a_7, b_8), (a_8, b_4), (a_8, b_8), (a_8, b_9), (a_9, b_9), (a_9, b_{10}), (a_{10}, b_6), (a_{10}, b_{10}). It is easily verified that K is irreducible and does not contain a cycle of rank 10. However, if S, T are partitioned $S = (A_1, A_2)$, $T = (B_1, B_2)$ where $A_1 = (a_1, a_2, a_3, a_4, a_5)$, $A_2 = (a_6, a_7, a_8, a_9, a_{10})$, $B_1 = (b_1, b_2, b_3, b_4, b_5)$, $B_2 = (b_6, b_7, b_8, b_9, b_{10})$ then the induced graph is of exterior dimension 2 and does contain a cycle of rank 2.

The following theorems show how the general irreducible graph is constructed.

THEOREM 3. *Let K be a finite graph and S and T be its vertex sets. Suppose there is a partitioning of S and T into disjoint subsets $S = A_1 \cup A_2 \cup \ldots \cup A_r$, $T = B_1 \cup B_2 \cup \ldots \cup B_r$ such that each of the subgraphs $K \cap (A_i \times B_i)$ $(i = 1, 2, \ldots, r)$ is irreducible. Let K^* be the graph induced by the partition. Then K is irreducible if and only if K^* is irreducible.*

Proof. Let $[A, B]$ be an m.e.p. for the graph K. Let $a \in A$. Then $a \in A_i$ for some i. Since the only m.e.p.'s for $K \cap (A_i \times B_i)$ are $[A_i, \phi]$ and $[\phi, B_i]$, this implies that every $b \in A_i$ is contained in A. Hence A consists of the union of some of the subsets $A_1, A_2, \ldots, A_r$. Similarly B consists of

the union of some of the subsets $B_1, B_2, \ldots, B_r$. Hence $[A, B]$ induces a covering of K^*. On the other hand, every m.e.p. $[A^*, B^*]$ of K^* induces the covering $[A, B]$ of K where A consists of the union of the elements in the subsets of S which make up A^* and B consists of the union of the elements in the subsets of T which make up $B.^*$ Hence, there is a one-one correspondence of the m.e.p.'s of K and of K^*, from which the theorem follows.

The way in which an irreducible graph is constructed from simple irreducible graphs can now be described. Incidentally, this process gives a rapid method of determining whether a given graph is or is not reducible. Let K be a graph with vertex sets S and T such that $\nu(S) = \nu(T) = n$, and let $E(K) = n$. Then K contains at least one diagonal. Putting $S = (a_1, a_2, \ldots, a_n)$ and $T = (b_1, b_2, \ldots, b_n)$ we may assume that one diagonal consists of the edges (a_i, b_i) $(i = 1, 2, \ldots, n)$. Call this diagonal the main diagonal. If K contains an edge which is alone in its row or column it is reducible. In other cases, a cycle of K half of whose edges come from the main diagonal can be determined as follows: Let (a_{i_1}, b_{j_1}) be any edge of K, not on the main diagonal. Then K contains the edge (a_{i_1}, b_{i_1}) which is on the main diagonal and in the same row, and the edge (a_{i_2}, b_{i_1}) which is in the same column, and the edge (a_{i_2}, b_{i_2}) etc. Every alternate edge is on the main diagonal. After a finite number of steps we arrive at a first edge which is repeated. The required cycle is obtained by starting with the first occurrence of this edge and including all edges of the sequence following it until the second occurrence. If the cycle is of rank n, then K is irreducible and the process stops. If the cycle obtained is of rank $r < n$ (it is still possible that K may contain a cycle of rank n) let A_1 and B_1 be the vertex sets of the cycle. The region diagonally opposite $A_1 \times B_1$ namely $\bar{A}_1 \times \bar{B}_1$ contains $n - r$ edges of the main diagonal of K. Since $\nu(\bar{A}_1) = \nu(\bar{B}_1) = n - r$, the subgraph $K \cap (\bar{A}_1 \times \bar{B}_1)$ has exterior dimension $n - r$. If this subgraph has at least two edges in each of its rows and columns we proceed in the same manner to extract a cycle half of whose edges come from the main diagonal. Let A_2 and B_2 be its vertex sets. The process is continued with the subgraph

$$K \cap (\overline{A_1 \cup A_2} \times \overline{B_1 \cup B_2})$$

etc. Ultimately either the entire vertex sets S and T are subdivided or a stage is reached where a subgraph

$$K \cap (\overline{A_1 \cup A_2 \ldots \cup A_s} \times \overline{B_1 \cup B_2 \ldots \cup B_s})$$

has edges which are alone in their row or column. Let

$$\nu\,(A_1 \cup A_2 \ldots \cup A_s) = t$$

and let

$$\overline{A_1 \cup A_2 \ldots \cup A_s}$$

consist of the vertices $a_{t+1}, a_{t+2}, \ldots, a_n$ and

$$\overline{B_1 \cup B_2 \ldots \cup B_s}$$

consist of the vertices $b_{t+1}, b_{t+2}, \ldots, b_n$. The vertex sets S and T are now partitioned into

$$S = A_1 \cup A_2 \cup \ldots \cup A_s \cup a_{t+1} \cup \ldots \cup a_n$$

$$T = B_1 \cup B_2 \ldots \cup B_s \cup b_{t+1} \cup \ldots \cup b_n.$$

Also, each of the subgraphs $K \cap (A_i \times B_i)$, $K \cap (a_i \times b_i)$ is irreducible. By Theorem 3, the graph K^* induced by this partition is irreducible if and only if K is irreducible. One then applies a similar partitioning to the vertex sets of K^*. After a finite number of such steps one arrives at a graph which is a simple irreducible graph in which case K is irreducible or one arrives at a graph which has an edge which is alone in its row or column in which case K is reducible.

5. The stability of the structure. A graph may be altered by the addition or the removal of edges. Such changes may alter the structure defined in §3. However, there are certain parts of the structure which remain unaltered by these changes. In **(3**, §5**)**, a first attempt was made to study these effects. In this section considerably more precise results are obtained which are listed below as properties.

In our decomposition of K in the form $K = C_1 + C_2 + \ldots + C_t$ with

$$C_i = \sum_j G_j^{(i)} \oplus \sum_p H_p^{(i)} \oplus \sum_q L_q^{(i)},$$

each of the subgraphs $G_j^{(i)}$, $H_p^{(i)}$, $L_q^{(i)}$ is contained in exactly one minimal square or rectangular block, namely, the Cartesian product of its vertex sets. These are called the blocks of the decomposition.

Property 1. If any number of edges are added to K in such a way that each added edge lies in a block of the decomposition, the block structure is unaltered. Each subgraph $G_j^{(i)}$ or $H_j^{(i)}$ or $L_q^{(i)}$ has added to it those edges which were added to its block. Property 1 follows from the fact that each of the subgraphs contained in the blocks is either irreducible or minimal semi-irreducible and adding edges to such subgraphs does not alter their status.

Property 2. Removal of edges from the cores $C_{i+1}, C_{i+2}, \ldots, C_t$ leaves unchanged the cores $C_1, C_2, \ldots, C_i$. This follows from the fact that the edges of $C_{i+1} + C_{i+2} \ldots + C_t$ form the inadmissible set of C_i.

Property 3. Removal of edges from a subgraph $G_i^{(1)}$ has the following effects:

(1) If the edges remaining form an irreducible graph $G_i^{(1)*}$ such that

$E(G_i^{(1)*}) = E(G_i^{(1)})$, the block structure of K is unchanged with the subgraph $G_i^{(1)}$, replaced by $G_i^{(1)*}$.

(2) If the edges remaining form a graph $G_i^{(1)*}$ which is such that $E(G_i^{(1)*}) = E(G_i^{(1)})$ but $G_i^{(1)*}$ is reducible then the following changes take place. The core of the altered graph K consists of the core of $G_i^{(1)*}$ together with all the remaining subgraphs $G_j^{(1)}$, $H_p^{(1)}$, $L_q^{(1)}$. The inadmissible set of $G_i^{(1)*}$ is added to the cores which come after C_1 and may alter their structure considerably.

(3) If $E(G_i^{(1)*}) < E(G_i^{(1)})$ the following takes place. All components of C_1 except $G_i^{(1)}$ still remain in C_1. Some of the edges of the remaining cores $C_2, C_3, \ldots$, may become admissible and enter C_1. In general the block structure of $C_2, C_3, \ldots$, may be destroyed.

Property 4. Removal of edges from a block $H_i^{(1)}$ (or $L_i^{(1)}$) has the following effect.

(1) If the resulting graph $H_i^{(1)*}$ is such that $E(H_i^{(1)*}) = E(H_i^{(1)})$ there are three possibilities:

If $H_i^{(1)*}$ is semi-irreducible then the block structure of K is unchanged, except that $H_i^{(1)}$ is replaced by $H_i^{(1)*}$ which is a disjoint sum of components of the type $H_p^{(1)}$.

If $H_i^{(1)*}$ is irreducible then it is a component of the type $G_p^{(1)}$.

If $H_i^{(1)*}$ is reducible, then the effect is the same as in case (2) of Property 1.

(2) If $E(H_i^{(1)*}) < E(H_i^{(1)})$ the effect is the same as in case (3) of Property 3.

Because of Property 2, the general case of removing edges from $G_i^{(j)}$, $H_i^{(j)}$, $L_i^{(j)}$, does not need special consideration. None of these operations affects the subgraph $C_1 + C_2 + \ldots + C_{j-1}$, and the effect on the rest of the graph is obtained from the fact that each of $G_i^{(j)}$, $H_i^{(j)}$, $L_i^{(j)}$ are components of the first core of $K - (C_1 + C_2 + \ldots + C_{j-1})$.

If exactly one edge is removed from a block of the core of a graph, then the exterior dimension is reduced only if the block is of the type $G_p^{(1)}$ and consists of a single edge, that is if $E(G_p^{(1)}) = 1$. Removal of exactly one edge from a block of the type $H_p^{(1)}$ or $L_p^{(1)}$ cannot change the exterior dimension.

Our next properties deal with the addition of edges which do not lie in the blocks of the decomposition of K. Let C_1 be the core of K and let R_1, R_2, R_3 be defined as in §2. For our present purposes we further break up R_3 and the regions $\bar{A}^* \times B_*$ and $A_* \times \bar{B}^*$ as follows. Using the same notation as in §2, we put

$$R_3 = R_3^\alpha + R_3^\beta + R_3^\gamma + R_3^\delta$$

$$R_3^\alpha = \cup_{i>j}(S_i \times T_j),\ R_3^\beta = \cup\,(S_i \times V_j) \text{ (taken over all } i \text{ and } j),$$

$$R_3^\gamma = \cup\,(X_i \times T_j) \text{ (taken over all } i \text{ and } j) \text{ and}$$

$$R_3^\delta = \cup\,(X_i \times V_j) \text{ (taken over all } i \text{ and } j).$$

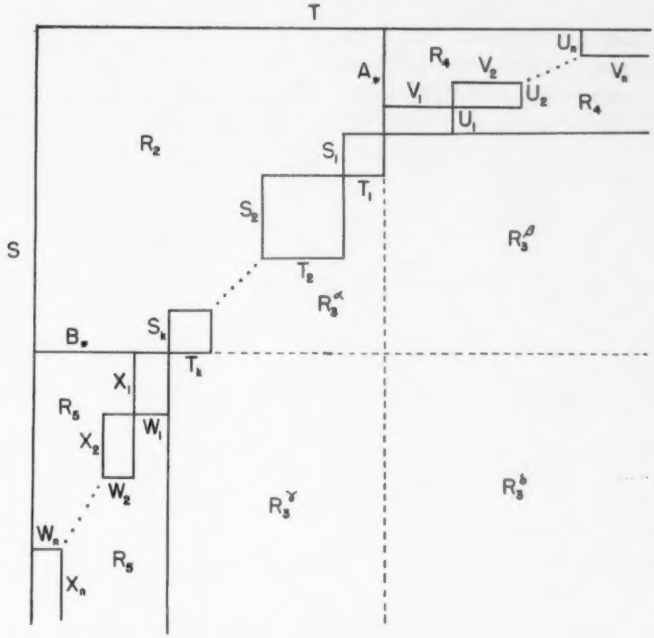

FIGURE 2.

Also let R_4 be the subregion of $A_* \times \bar{B}^*$ defined by

$$R_4 = \bigcup_{i \neq j} (U_i \times V_j).$$

Similarly let R_5 be the subregion of $\bar{A}^* \times B_*$ defined by

$$R_5 = \bigcup_{i \neq j} (X_i \times W_j).$$

None of the regions just defined contains any of the edges of K. Figure 2 gives a schematic representation of the various blocks.

Property 5. Addition of any number of edges to K in the region R_2 leaves the core C_1 unchanged. More generally, addition of any number of edges to a region R_2 which corresponds to the graph $K - (C_1 + C_2 + \ldots + C_r)$ leaves each of the cores $C_1, C_2, C_3, \ldots, C_r, C_{r-1}$ unchanged. This property follows from our definition of inadmissible edges and from Theorem 10 of (3).

The remaining properties deal with the effect of the addition of a single edge.

Property 6. The effect of adding to K an edge from R_4. Let (s, t) be the added edge. Then $s \in U_i$ for some i and $t \in V_j$ for some $j, j \neq i$. The sub-

graph consisting of the edge (s, t) together with $H_i^{(1)}$ and $H_j^{(1)}$ has as its only cover $[U_i \cup U_j, \phi]$. Also it cannot be decomposed into a disjoint sum. It therefore goes into C_1 as a component of type $H_r^{(1)}$ replacing the components $H_i^{(1)}$ and $H_j^{(1)}$. The rest of the decomposition of K is unaffected.

The addition of an edge from R_5 has a similar effect.

Property 7. The effect of adding to K an edge from R_3^α. Let (s, t) be the added edge. Consider the subgraph $K \cap \sum_i G_i^{(1)}$ with vertex sets M and N where $M = \cup S_i$, $N = \cup T_i$ and consider the subgraph K^* induced by the partition

$$M = S_1 \cup S_2 \cup \ldots \cup S_k, \quad N = T_1 \cup T_2 \cup \ldots \cup T_k.$$

If $(s, t) \in S_i \times T_j$, (S_i, T_j) is the edge added to K^* which is induced by the addition of (s, t) to K. Before this addition K^* did not have a cycle half of whose edges were on the main diagonal (the edges (S_1, T_1), (S_2, T_2), $\ldots$, (S_k, T_k)) since all edges of K^* are on or above the main diagonal. If after the addition K^* still contains no such cycle then (S_i, T_j) is an inadmissible edge of K^*. Hence (s, t) is an inadmissible edge of K by the proof of Theorem 3. Hence this addition does not affect the core C_1 but may change the structure of subsequent cores. On the other hand if the edge (S_i, T_j) is part of a cycle half of whose edges are from the main diagonal, let the cycle consist of (S_i, T_j) together with (S_1, T_1), $(S_2, T_2), \ldots$, (S_u, T_u), and $(u - 1)$ edges of the form (S_r, T_s) with $r < s$. Then the subgraph of K^* namely

$$K^* \cap \{(S_1 \cup S_2 \cup \ldots \cup S_u) \times (T_1 \cup T_2 \cup \ldots \cup T_u)\}$$

is irreducible. Hence by Theorem 3, the subgraph of the altered K consisting of the edge (s, t) together with the edges of

$$K \cap \{(S_1 \cup S_2 \cup \ldots \cup S_u) \times (T_1 \cup T_2 \cup \ldots \cup T_u)\}$$

form an irreducible subgraph which belongs to the core C_1. This subgraph replaces the set of subgraphs $G_1^{(1)}, G_2^{(1)}, \ldots, G_u^{(1)}$ and always contains edges which previously were inadmissible. The rest of C_1 is unaltered. However, the structure of the remaining cores is destroyed since some of their edges have entered C_1.

Property 8. The effect of adding an edge from R_3^β to K. Let (s, t) be the added edge. Then $s \in S_i$, $t \in V_j$ for some i and j. Consider the graph K^* induced from K by the subpartitioning

$$S_i \cup S_{i-1} \cup \ldots \cup S_1 \cup U_1 \cup U_2 \cup \ldots \cup U_n$$

and

$$T_i \cup T_{i-1} \cup \ldots \cup T_1 \cup V_1 \cup V_2 \cup \ldots \cup V_n.$$

Call the diagonal given by the edges (S_i, T_i), $(S_{i-1}, T_{i-1}), \ldots, (S_1, T_1)$, (U_1, V_1), $(U_2, V_2), \ldots, (U_n, V_n)$ the main diagonal of K^*. Consider a sub-

graph of K^* consisting of $2p+1$ edges of which the first is (S_i, T_i), the second is above (S_i, T_i) in the same column, the third is to the right of the second in the same row on the main diagonal etc., moving up and to the right each time. Such a graph contains $p+1$ edges on the main diagonal. Let K^*_1 be the union of all such subgraphs of K^*, and let $K_1 = \cup(P \times Q)$, the union being taken over all edges (P, Q) of K^*_1. The subgraph $K \cap [K_1 \cup_j (U_j \times V_j) \cup (s, t)]$ is a minimal semi-irreducible graph and is a component of the type $H_p^{(1)}$. Thus any edge (P, Q) of K^*_1 which is not on the main diagonal corresponds to a block $P \times Q$ which is inadmissible in K but is admissible after the addition of (s, t). Blocks of the graph other than $(U_j \times V_j)$ and those which make up K_1, are unchanged by the addition of (s, t). In general the structure of cores subsequent to C_1 is destroyed.

Property 9. The effect of adding an edge from R_3^5 to K. First to be noted is that the region R_3^5 is the only region in which the addition of an edge increases the value of $E(K)$. Let (s, t) be the added edge. Then $s \in X_i$, $t \in V_j$ for some i and j. Consider the subgraphs of the modified K consisting of $L_i^{(1)} \cup H_j^{(1)} \cup (s, t)$. Its only m.e.p.'s are $[U_j \cup s, W_i]$ and $[U_j, W_i \cup t]$. This implies that in the modified graph, the edges in the same row as s (except (s, t)) and the edges in the same column as t (except (s, t)) are inadmissible and hence are to be omitted from the first core. After deleting all the inadmissible edges from the modified graph the edge (s, t) is alone in its row and column and hence forms a one by one irreducible block which belongs to the core C_1. Hence (s, t) becomes a block of form $G_p^{(1)}$. The subgraph $H_j^{(1)}$ has one column of its edges removed (those edges in the same column as t). If this modified $H_j^{(1)}$ is semi-irreducible or irreducible it remains a component of C_1 of type $H_p^{(1)}$ or $G_p^{(1)}$. If the modified $H_j^{(1)}$ is reducible its core enters C_1 and its remaining edges become inadmissible. The subgraph $L_i^{(1)}$ suffers a similar fate. Finally, in general the structure of cores subsequent to C_1 becomes destroyed.

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SECTION III

Similarities Over Fields of Characteristic Two

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Let E, E' denote finite dimensional vector spaces over a field of characteristic $\neq 2$. In either space a symmetric bilinear form (x, y) is given. Let T be a linear transformation of E into E'. We may call T a—possibly degenerate—similarity if it has one of the following three properties: (i) There exists a ρ such that $(Tx, Ty) = \rho \, . \, (x, y)$ for all x, y (if $\rho \neq 0$ this can be interpreted as the invariance of angles). (ii) $(x, y) = 0$ always implies $(Tx, Ty) = 0$ ("orthogonality" is preserved). (iii) $(x, x) = 0$ always implies $(Tx, Tx) = 0$ ("isotropy" is preserved). Obviously (i) $\rightarrow$ (ii) $\rightarrow$ (iii). The equivalence of (i) and (ii) is not hard to prove. It can be proved that (iii) $\rightarrow$ (i) if E contains vectors $x \neq 0$ with $(x, x) = 0$ and if either the field has more than three elements or T is a mapping of E onto itself. In a letter to the author, Professor Dieudonné kindly pointed out that these results were fairly well known but that the corresponding problems for characteristic 2 did not seem to have been discussed.

Let E, E' now denote finite dimensional vector spaces over a field K of characteristic 2. In order to have an analytic geometry it is not sufficient that bilinear forms are given. We have to assume the existence of a quadratic form $[x]$ in either space. A linear transformation T of E into E' may then be called a similarity if either (i) $[Tx] = \rho \, . \, [x]$ for some ρ and every $x \in E$ (if $\rho \neq 0$, we may say that ratios of "squares of lengths" are preserved) or (ii) $[x] = 0$ always implies $[Tx] = 0$ ("singularity" is preserved).

Clearly (i) implies (ii) but not vice versa. Our main result (Theorem 2) states that (ii) $\rightarrow$ (i) if K is not the prime field $F_2(bf_1)$ of characteristic 2 and if a certain weak additional condition is satisfied. In the last sections some remarks on this condition and on the case $K = F_2(bf_1)$ are added.

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1. In this paper K denotes a commutative field of characteristic 2. Greek letters denote elements of K. Let E be a finite dimensional vector space over K. The mapping $x \rightarrow [x]$ of E into K is called a quadratic form in E if

(1) $$[\lambda x] = \lambda^2[x] \qquad \text{for every } \lambda \in K,\ x \in E$$

and if

(2) $$(x, y) = [x + y] + [x] + [y]$$

is a bilinear form. This definition implies $[0] = 0$,

(3) $[\lambda x + \mu y] = \lambda^2[x] + \mu^2[y] + \lambda\mu(x, y)$ for every $\lambda, \mu \in K$, $x, y \in E$, and

(4) $(x, y) = (y, x) = -(y, x)$ in particular $(x, x) = 0$ for every $x, y \in E$.

If $(x, y) = 0$, x and y may be called orthogonal. The set of all the vectors which are orthogonal to every vector of E is a subspace E^* of E. By (4),

$$\operatorname{rank} E = \dim E - \dim E^*$$

is an even number. If $x \notin E^*$, the vectors orthogonal to x form a subspace of E of dimension $\dim E - 1$.

The subspace V of E is called isotropic if some vector $\neq 0$ of V is orthogonal to every vector of V. Thus V must be isotropic if $\dim V$ is odd. If the form (x, y) vanishes identically in V, V is called totally isotropic.

We call the vector a singular if $[a] = 0$. The subspace V is singular if it contains a singular vector $a \neq 0$ which is orthogonal to every vector of V. If $[x]$ vanishes identically in V, V is called totally singular. The index ν of the form $[x]$ is the maximum dimension of all the totally singular subspaces of E.

Let E_0 denote the subspace of all the singular vectors in E^*. The vector space $\bar{E} = E/E_0$ consists of the classes of vectors $\bar{x} = x + E_0$. Thus the null-vector $\bar{0}$ of $\bar{E}$ is the class E_0. We define the quadratic form $[\bar{x}]$ in $\bar{E}$ uniquely through $[\bar{x}] = [x]$ and readily verify $\bar{E}^* = E^*/E_0$ and $\bar{E}_0 = \bar{0}$. If $\bar{\nu}$ is the index of the form $[\bar{x}]$, then

(5) $$\bar{\nu} = \nu - \dim E_0.$$

Any totally singular subspace of E of maximal dimension contains E_0.

2. Suppose a second vector space over K is given with a quadratic form in it. We denote it and its bilinear form again by $[x]$ and (x, y) respectively.

Let T be a linear transformation of E into the second space. Then $[Tx]$ can be reinterpreted as a quadratic form in E. The bilinear form associated with it obviously is (Tx, Ty). This suggests the notation

$$[Tx] = [x]^T, \qquad (Tx, Ty) = (x, y)^T.$$

Thus we can also interpret T as the transition from one quadratic form $[x]$ in E to another quadratic form $[x]^T$ in the same space.

We wish to study the following properties of T:

(I) There is a ρ such that $[x]^T = \rho[x]$ for every $x \in E$.
(II) $(x, y)^T = \rho \,.\, (x, y)$ for every $x, y \in E$.
(III) $[x] = 0$ always implies $[x]^T = 0$.
(IV) $(x, y) = 0$ always implies $(x, y)^T = 0$.

3. Obviously, (II) and (III) follow from (I) and (IV) is a consequence of (II). Clearly, (III) does not follow from (II): Let $a_1, \ldots, a_n$ be a basis of E. The form $[x]$ is defined if the $[a_i]$ and (a_i, a_k) are prescribed so that (4) is satisfied. Choose all the (a_i, a_k) equal to one another while not all the $[a_i]$ are equal $[i, k = 1, 2, \ldots, n;\ i \neq k]$. A suitable permutation of the vectors $a_1, \ldots, a_n$ defines a linear transformation T of E onto itself which satisfies (II) with $\rho = 1$ but not (III).

It may be known that (II) and (IV) actually are equivalent. We include a proof for the reader's convenience.

THEOREM 1. (IV) *implies* (II).

The proof is based on two lemmas.

LEMMA 1. *Suppose Theorem* 1 *holds true if* $\dim E < n$. *Let* $\dim E = n$, rank $E < n$. *If T satisfies* (IV), (II) *will hold in E.*

Proof. Let $a \neq 0$, $a \in E^*$. Decompose E into the direct sum of the straight line through a and an $(n-1)$ – space E'. Thus every $x \in E$ permits a decomposition $x = x' + \xi . a$ where $x' \in E'$. Let $y = y' + \eta a$ be a second vector; $y' \in E'$. Then $(a, x') = (a, y') = 0$ and therefore $(a, x')^T = (a, y')^T = 0$. Hence by our assumptions

$$(x, y)^T = (x', y')^T = \rho . (x', y') = \rho . (x, y).$$

LEMMA 2. *Let $a_1, \ldots, a_n$ be a basis of E. Suppose there is a ρ such that $(a_i, a_k)^T = \rho . (a_i, a_k)$ for every i, k. Then* (II) *holds.*

The proof is obvious.

Proof of Theorem 1. If $\dim E = 2$, the assumptions of Lemma 2 are trivially satisfied. If $\dim E = 3$, then rank $E < 3$ and we may apply Lemma 1 with $n = 3$.

Let $\dim E = 4$. On account of Lemma 1 we may assume rank $E = 4$. Let a, b be any two linearly independent vectors; $(a, b) \neq 0$. The vectors orthogonal to a form a three-space E' spanned, say, by a, c', d'. Put

$$c = c' + \frac{(c', b)}{(a, b)} \cdot a, \qquad d = d' + \frac{(d', b)}{(a, b)} \cdot a.$$

Then

$$(a, c) = (a, d) = (b, c) = (b, d) = 0.$$

Hence by (IV)

$$(a, c)^T = (a, d)^T = (b, c)^T = (b, d)^T = 0. \tag{6}$$

Obviously a, c, d span E'. Thus a, b, c, d is a basis of E. Since rank $E = 4$,

this implies $(c, d) \neq 0$. Replacing b and d by suitable multiples we may assume

$$(a, b) = (c, d) = 1.$$

Since $(a + c, b + d) = (a, b) + (c, d) = 0$, (IV) implies $(a + c, b + d)^T = 0$. Hence by (6)

$$(a, b)^T = (c, d)^T = \rho.$$

By Lemma 2, (II) will hold with this ρ.

Finally let dim $E > 4$. We may assume E is not totally isotropic. Let a, b, x, y be any four vectors with $(a, b) \neq 0$, $(x, y) \neq 0$. They span a subspace of dimension $\leqslant 4$. From the above

$$(x, y)^T/(x, y) = (a, b)^T/(a, b).$$

This completes our proof.

4. We now show that (I) can be derived from (III) under two additional assumptions. $F_2(bf_1)$ denoted the prime field with two elements.

THEOREM 2. *Let* $K \neq F_2(bf_1)$ *and let* $\bar{\nu} > 0$; cf. **(1)** *Then* (III) *implies* (I) (*and hence also* [II] *and* [IV]).

Proof. We first note that by (5) there exists a singular vector $a \notin E^*$. Thus there is a vector d such that $(a, d) \neq 0$. Put

$$b = ([d]\, a + (a, d) \,.\, d)/(a, d)^2.$$

Then

$$[a] = [b] = 0; \ (a, b) = 1. \tag{7}$$

By (III)

$$[a]^T = [b]^T = 0. \tag{8}$$

We wish to show

$$[x]^T = (a, b)^T \,.\, [x] \qquad \text{for all } x \in E. \tag{9}$$

If $x = \alpha a + \beta b$, then by (7) $[x] = \alpha\beta$, hence by (8)

$$[x]^T = [\alpha a + \beta b]^T = a\beta(a, b)^T = (a, b)^T \,.\, [x].$$

From now on let a, b, x be linearly independent. Thus they span a three-space V. Put

$$c = x + (b, x)\, a + (a, x) b. \tag{10}$$

Then a, b, c is again a basis of V and

$$(a, c) = (b, c) = 0. \tag{11}$$

Our next goal is the proof of

$$(a, c)^T = (b, c)^T = 0. \tag{12}$$

For every λ we have

$$[[c] \,.\, a + \lambda \,.\, c + \lambda^2 \,.\, b] = 0.$$

Hence by (III)

$$[[c] \,.\, a + \lambda \,.\, c + \lambda^2 \,.\, b]^T = 0$$

or on account of (8),

$$\lambda \,.\, [c] \,.\, (a, c)^T + \lambda^2([c] \,.\, (a, b)^T + [c]^T) + \lambda^3 \,.\, (b, c)^T = 0.$$

This is a third degree polynomial in λ with coefficients in K which vanishes for every $\lambda \in K$. Since $K \neq F_2(bf_1)$, this polynomial has more than three roots. Hence each coefficient vanishes:

$$[c] \,.\, (a, c)^T = [c] \,.\, (a, b)^T + [c]^T = (b, c)^T = 0. \tag{13}$$

If $[c] \neq 0$, this yields (12).

If $[c] = 0$, then $[c]^T = 0$. In this case $[a + c] = 0$. Hence $[a + c]^T = 0$ and $(a, c)^T = 0$. This with (13) again yields (12).

Let $x = \alpha a + \beta b + c$; cf. (10). Then (8), (12), (13), (7), and (11) imply

$$\begin{aligned} [x^T] = [\alpha a + \beta b + c]^T &= [c]^T + \alpha\beta \,.\, (a, b)^T \\ &= (a, b)^T \,.\, ([c] + \alpha\beta) \\ &= (a, b)^T \,.\, [\alpha a + \beta b + c]. \end{aligned}$$

This proves (9) and our theorem.

5. If $E = E_0$, E will be totally singular. (III) then implies that $[x]^T$ also vanishes identically and (I) becomes trivial.

Let $E \neq E_0$ and $\bar{v} = 0$. Thus $[x] = 0$ if and only if $x \in E_0$. Decompose E into the direct sum of E_0 and a subspace E' and let x_0 and x' range through E_0 and E' respectively. Thus $[x_0 + x'] = [x']$ and $(x_0, x') = 0$. Choose any bilinear form $(x_0, x')^T$ which does not vanish identically and any quadratic form $[x']^T$ in E'. Then

$$[x_0 + x']^T = (x_0, x')^T + [x']^T$$

defines a quadratic form $[x]^T$ in E which satisfies (III) but not (IV) and therefore neither (I) nor (II).

6. Let K be perfect and let $\dim E > 2$. Then there are two linearly independent vectors a, b such that $(a, b) = 0$. If $[a] \neq 0$ and $[b] \neq 0$, then

$$[a/\sqrt{[a]} + b/\sqrt{[b]}] = [a/\sqrt{[a]}] + [b/\sqrt{[b]}] = 1 + 1 = 0.$$

Hence E always contains singular vectors $\neq 0$, that is, $\nu > 0$.

Applying the preceding remark to $\bar{E}$ instead of E, we obtain the following commentary to Theorem 2: If K is perfect and dim $\bar{E} > 2$, then $\bar{\nu} > 0$.

7. In the case $K = F_2(Fbf_1)$, (III) and $\bar{\nu} > 0$ or (III) and (IV) are readily seen not to imply (I). However, it can be shown that (I) follows from (III), (IV), and $\bar{\nu} > 0$. We shall merely prove: Let E be a vector space over $F_2(bf_1)$ and let T be a linear transformation of E *onto itself* which satisfies (III). Then (I) will hold.

Proof. T is a one-one mapping of the finite set E onto itself. As it maps the subset of the singular vectors onto itself, it must also map its complement onto itself. But this complement consists of the vectors x with $[x] = 1$. Thus $[x]^T = [x]$ for every x.

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SECTION III

Extensions of Wirtinger's Inequality

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Presented by G. F. D. DUFF, F.R.S.C.

1. Introduction. In this paper we deal with several extensions of the following theorem of Wirtinger, a concise proof of which may be found in **(3**, Theorem 258**)**.

If $u' \in L_2$, $u(-\pi) = u(\pi)$, *and* $\int_{-\pi}^{\pi} u\,dx = 0$, *then*

$$\int_{-\pi}^{\pi} u^2\,dx < \int_{-\pi}^{\pi} u'^2\,dx, \text{ unless } u = A\sin(x+\alpha). \tag{1.1}$$

Moreover, this inequality is sharp, that is, the unit constant factor on the right side cannot be decreased.

More than fifteen years ago, Bellman **(2)** proved the following extension of (1.1):

If $u^{(n)} \in L_{2k}$, $u(x) \equiv u(x+2\pi)$, *and* $\int_{-\pi}^{\pi} u\,dx = 0$, *then*

$$\int_{-\pi}^{\pi} u^{2k}\,dx \leqslant a_n^{2k} \int_{-\pi}^{\pi} [u^{(n)}]^{2k}\,dx, \tag{1.2}$$

where k, n *are positive integers and the* a_n *are certain constants, for example,* $a_1 = \pi/2$, $a_2 = \pi^2/8$.

For $k = n = 1$, (1.2) is weaker than (1.1); also (1.2) is not sharp for $k = 1$, $n = 2$ (cf. **(1**, Theorem 2.6**)**). In **(1)**, the inequality (1.1) was generalized to inequalities of the form

$$\int_a^b s(x)u^2\,dx \leqslant \int_a^b u'^2\,dx, \tag{1.3}$$

$$\int_a^b s(x)u^2\,dx \leqslant \int_a^b u''^2\,dx, \tag{1.4}$$

where u is subject to certain boundary conditions, and a condition of the form

$$\int_a^b s\,u\,dx = 0.$$

This paper deals with integral inequalities of the form

$$\int_a^b s(x)u^{2k}\,dx \leqslant \int_a^b r(x)u'^{2k}\,dx, \tag{1.5}$$

where u is again subject to certain boundary conditions, and

$$\int_a^b s\,u^{2k-1}dx = 0.$$

The inequality (1.5) thus subsumes the inequality (1.3), but differs from the case $n = 1$ of (1.2) in the side condition (except when $k = 1$). In fact, the case $s \equiv 1$, $r \equiv$ constant, $a = -\pi$, $b = \pi$ of (1.5) requires a separate treatment from the main result (1.5). In §2, we deal with the general result (1.5), and discuss the sharpness of this inequality; we conclude this section by giving several examples. The modifications needed for the special extension of Wirtinger's inequality are then handled in §3.

2. The general case. For brevity, we shall write $2k = p$, and $q = p/(p-1)$, so that q, p are conjugate exponents for Hölder's inequality. In the nonlinear differential equation

$$\frac{d}{dx}\{r(x)y'^{p-1}\} + s(x)y^{p-1} = 0, \tag{2.1}$$

we shall always assume that $r(x)$, $r'(x)$, $s(x)$ are continuous, with $r(x) > 0$, $s(x) \geqslant 0$ on an interval $a < x < b$, except that $r(x)$ may have a single zero, or a single discontinuity at a point $x = \bar{x}$, $a < \bar{x} < b$. Here, a or b (or both) may be infinite. We say that a function $u(x)$ is an *integral* on (a, b) if, for any $c \in (a, b)$, we have

$$u(x) = u(c) + \int_c^x u'(t)dt, \qquad a < x < b.$$

THEOREM 2.1. *Let $p = 2k$ and assume that the differential equation (2.1) has a solution $y(x)$ which is an integral on (a, b), and that $y(x)$ is negative for $a < x < \bar{x}$, and positive for $\bar{x} < x < b$. In addition, we assume that*

$$\frac{y'(x)}{y(x)} = O[(x-\bar{x})^{-1}],\ \frac{y'(x)}{y(x)} = O[(x-a)^{-1}],\ \frac{y'(x)}{y(x)} = O[(b-x)^{-1}] \tag{2.2}$$

for x near $\bar{x}$, a, b respectively, and that $r(x)$ satisfies the three conditions

$$r(x) = O[(x-\bar{x})^{p-1}],\ \text{or}\ r^{q/p}(x)\int_{\bar{x}}^x r^{-q/p}(t)dt = O(x-\bar{x}), \tag{2.3}$$

$$r(x) = o[(x-a)^{p-1}],\ \text{and}\ r^{q/p}(x)\int_x^{k_1} r^{-q/p}(t)dt = O(x-a), \tag{2.4}$$

$$r(x) = o[(b-x)^{p-1}],\ \text{and}\ r^{q/p}(x)\int_{k_2}^x r^{-q/p}(t)dt = O(b-x), \tag{2.5}$$

for x near $\bar{x}$, a, b respectively. In (2.4), (2.5), *k_1 and k_2 are any constants such that $k_1 \leqslant \bar{x} \leqslant k_2$.*

Now, let $u(x)$ be an integral on (a, b), and suppose that

$$\int_a^b r\, u'^p\, dx < \infty, \qquad \int_a^b s\, u^{p-1}\, dx = 0. \tag{2.6}$$

Then

$$\int_a^b s\, u^p\, dx \leqslant \int_a^b r\, u'^p\, dx. \tag{2.7}$$

Moreover, equality holds in (2.7) *only if $u(x) \equiv c\, y(x)$, where $c = 0$ unless*

$$\int_a^b r\, y'^p\, dx < \infty.$$

Proof. The proof of this theorem is based on the elementary inequality

$$x^p + (p-1)y^p - pxy^{p-1} \geqslant 0, \ (p = 2k) \tag{2.8}$$

valid for all real x, y (**3**, Theorem 41**)**. Equality holds only if $x = y$. Setting $x = a - b$, $y = a$, this inequality can also be written in the form

$$(a-b)^p > a^p - pba^{p-1}, \text{ unless } b = 0. \tag{2.9}$$

Our hypotheses on $r(x)$ stated after (2.1) ensure that $y'(x)$ is continuous on the subintervals $(a, \bar{x})$ and $(\bar{x}, b)$. Now set $u(\bar{x}) = \bar{u}$, and take $x = u'$, $y = [h(u - \bar{u})]^{1/(p-1)}$ in (2.8), where $h \equiv (y'/y)^{p-1}$. We then have, for any $[a', b'] \subset (a, b)$, and any $\epsilon > 0$

$$\begin{aligned} 0 \leqslant \int_{a'}^{\bar{x}-\epsilon} & r\{u'^p + (p-1)h^{p/(p-1)}(u-\bar{u})^p - phu'(u-\bar{u})^{p-1}\}\, dx \\ &+ \int_{\bar{x}+\epsilon}^{b'} r\{u'^p + (p-1)h^{p/(p-1)}(u-\bar{u})^p - phu'(u-\bar{u})^{p-1}\}\, dx, \end{aligned} \tag{2.10}$$

with equality only if $u'/(u - \bar{u}) \equiv y'/y$, that is, only if $u(x) - u(\bar{x})$ is a constant multiple of $y(x)$ on the subintervals $(a', \bar{x} - \epsilon)$ and $(\bar{x} + \epsilon, b')$. Integrating by parts the last term in each of the integrals of (2.10) we obtain

$$\begin{aligned} 0 \leqslant & \int_{a'}^{\bar{x}-\epsilon} ru'^p dx + (p-1)\int_{a'}^{\bar{x}-\epsilon} rh^{p/(p-1)}(u-\bar{u})^p dx + \int_{a'}^{\bar{x}-\epsilon} (rh)'(u-\bar{u})^p dx \\ & + \int_{\bar{x}+\epsilon}^{b'} ru'^p dx + (p-1)\int_{\bar{x}+\epsilon}^{b'} rh^{p/(p-1)}(u-\bar{u})^p dx + \int_{\bar{x}+\epsilon}^{b'} (rh)'(u-\bar{u})^p dx \\ & + rh(u-\bar{u})^p \Big|_{\bar{x}-\epsilon}^{\bar{x}+\epsilon} - rh(u-\bar{u})^p \Big|_{a'}^{b'}. \end{aligned}$$

It follows from (2.1) that h satisfies the Riccati-like equation

$$(rh)' + (p-1)rh^{p/(p-1)} = -s(x), \ (x \neq \bar{x}). \tag{2.11}$$

Hence the last inequality may be rewritten as

$$\int_{a'}^{\bar{x}-\epsilon} s(u-\bar{u})^p dx + \int_{\bar{x}+\epsilon}^{b'} s(u-\bar{u})^p dx$$
$$\leqslant \int_{a'}^{\bar{x}-\epsilon} ru'^p dx + \int_{\bar{x}+\epsilon}^{b'} ru'^p dx + rh(u-\bar{u})^p \Big|_{\bar{x}-\epsilon}^{\bar{x}+\epsilon} - rh(u-\bar{u})^p \Big|_{a'}^{b'}.$$

We now prove that

$$\lim_{x\to\bar{x}} rh(u-\bar{u})^p = 0, \tag{2.12}$$

whence the preceding inequality can be written as

$$\int_{a'}^{b'} s(u-\bar{u})^p dx \leqslant \int_{a'}^{b'} ru'^p dx - rh(u-\bar{u})^p \Big|_{a'}^{b'}. \tag{2.13}$$

Since $u - \bar{u} = o(1)$ as $x \to \bar{x}$, and $h = O[(x-\bar{x})^{1-p}]$, (2.12) is clearly satisfied if the first of conditions (2.3) holds. If the alternative holds, then we have

$$|u(x) - u(\bar{x})| \leqslant \int_{\bar{x}}^{x} |u'| dt \leqslant \left(\int_{\bar{x}}^{x} ru'^p dt\right)^{1/p} \left(\int_{\bar{x}}^{x} r^{-q/p} dt\right)^{1/q}$$

by Hölder's inequality. Hence

$$r(u-\bar{u})^p \leqslant \int_{\bar{x}}^{x} ru'^p dt \cdot \left\{ r^{q/p}(x) \int_{\bar{x}}^{x} r^{-q/p}\, dt \right\}^{p/q} = o(1)O[(x-\bar{x})^{p-1}]$$

so that (2.12) is again valid.

In the inequality (2.9), take $a = u$, $b = \bar{u}$; using the fact that $s(x) \geqslant 0$, (2.13) gives

$$\int_{a'}^{b'} su^p dx \leqslant \int_{a'}^{b'} ru'^p dx + pu(\bar{x}) \int_{a'}^{b'} su^{p-1} dx - rh(u-\bar{u})^p \Big|_{a'}^{b'}. \tag{2.14}$$

Here, equality can hold only if $u(\bar{x}) = 0$ and $u(x) - u(\bar{x}) = u(x)$ is a constant multiple of $y(x)$ on each of the two subintervals where $y(x) \neq 0$. Moreover, if inequality holds for any $[a', b'] \subset (a, b)$, then inequality will also hold in the limit as $a' \to a$ and $b' \to b$ since $u(\bar{x})$ is fixed, and since the right side of (2.10) is non decreasing as $[a', b']$ expands. Letting $a' \to a$ and $b' \to b$, the inequality (2.7) follows from (2.6) and (2.14), and

$$\lim_{x\to a} rh(u-\bar{u})^p = \lim_{x\to b} rh(u-\bar{u})^p = 0, \tag{2.15}$$

which we now prove. The first of (2.15) will follow from (2.4). Indeed, we have

$$r^{q/p}(x) \int_{x}^{k_1} r^{-q/p} dt \leqslant K(x-a)$$

for x near a Also, given $\epsilon > 0$ there corresponds $X \leqslant k_1$ such that

$$\left(\int_{a}^{X} ru'^p dt\right)^{1/p} K^{1/q} < \epsilon.$$

Hence

$$|u(x)| - |u(X)| \leqslant \int_x^X |u'|dt \leqslant \left(\int_x^X ru'^p dt\right)^{1/p}\left(\int_x^X r^{-q/p}dt\right)^{1/q},$$

$$r^{1/p}(x)|u(x)| \leqslant r^{1/p}(x)|u(X)| + \left(\int_a^X ru'^p dt\right)^{1/p}\left(r^{q/p}(x)\int_x^X r^{-q/p}dt\right)^{1/q}$$

$$< r^{1/p}(x)|u(X)| + \epsilon(x-a)^{1/q}.$$

Thus,

$$\frac{r^{1/p}(x)|u(x)|}{(x-a)^{1/q}} \leqslant \epsilon + \frac{r^{1/p}(x)}{(x-a)^{1/q}}|u(X)| = \epsilon + 0(1),$$

using the first of conditions (2.4). It follows that

$$r(x)u^p(x) = o[(x-a)^{p-1}]. \tag{2.16}$$

From (2.16) it also follows that

$$u^{p-1}(x) = r^{-1/q}(x)\, o[(x-a)^{p+p^{-1}-2}],$$

and

$$rhu^{p-1} = r^{1/p}(x)\, o[(x-a)^{p^{-1}-1}] = o(1), \tag{2.17}$$

where we again used the first of conditions (2.4). Finally, we use the fact that for $p \geqslant 2$, the inequality (2.8) may be improved to

$$(x-y)^p \leqslant x^p + (p-1)y^p - pxy^{p-1}, \quad (p = 2k) \tag{2.18}$$

Taking $x = \bar{u}$, $y = u$ we have

$$rh(u-\bar{u})^p \leqslant rh\bar{u}^p + (p-1)rhu^p - p\bar{u}rhu^{p-1} = o(1),$$

establishing the first part of (2.15). The second part follows in the same way, using (2.5). We may also note (2.17) implies that $y(x)$ is admissible if

$$\int_a^b ry'^p dx < \infty,$$

since we may then take $u(x) = y(x)$ in (2.17) giving

$$ry'^{p-1} = o(1)$$

for x near a or b. But then

$$\int_a^b sy^{p-1}dx = -ry'^{p-1}\Big|_a^b = 0, \tag{2.19}$$

so that $y(x)$ is admissible.

As for the possibility of equality in (2.7), assuming

$$\int_a^b ry'^p dx < \infty,$$

suppose

$$u(x) = \begin{cases} c_1 y(x), & a < x \leqslant \bar{x}, \\ c_2 y(x), & \bar{x} \leqslant x < b. \end{cases}$$

Using the second of conditions (2.6), and (2.19), we see that $c_1 = c_2$. Hence, noting the remarks following (2.14), the assertion concerning equality in (2.7) is verified, and the theorem is proved.

If y is admissible, the inequality (2.7) is sharp. *Suppose $s(x)$ is symmetric about $x = \bar{x}$ (and $\bar{x} = (a + b)/2$), and that $y(x)$ is antisymmetric about $\bar{x}$;* for convenience, we write $\bar{x} = 0$, $a = -x_1$, $b = x_1$ $(x_1 > 0)$, so we are assuming $s(-x) \equiv s(x)$ and $y(-x) \equiv -y(x)$. *Under these hypotheses* (2.7) *is sharp even if*

$$\int_a^b r y'^p dx = \infty,$$

provided that

$$\int_{-x_1}^{x_1} s dx < \infty, \; r y y'^{p-1} = o(1) \quad \text{as} \quad x \to \pm x_1. \tag{2.20}$$

To prove this, note first that ry'^p is integrable over $\bar{x} = 0$; for, from (2.1) we have

$$y(r y'^{p-1})' + s y^p \equiv 0,$$

whence

$$r y y'^{p-1} \Big|_\epsilon^x - \int_\epsilon^x r y'^p dt + \int_\epsilon^x s y^p dt \equiv 0.$$

Since $s(x)$, $y(x)$ are continuous at $x = 0$ and both integrands are positive, the existence of

$$\int_0 r y'^p dx$$

follows from the existence of

$$\lim_{\epsilon \to 0} r y y'^{p-1} = 0.$$

Now define an admissible function $u(x)$ by

$$u(x) = \begin{cases} y(-x_2), & -x_1 \leqslant x \leqslant -x_2, \\ y(x), & -x_2 \leqslant x \leqslant x_2, \\ y(x_2), & x_2 \leqslant x \leqslant x_1, \end{cases}$$

where x_2 will be assigned later. This function is certainly an integral on $(-x_1, x_1)$, and

$$\int_{-x_1}^{x_1} r u'^p dx = \int_{-x_2}^{x_2} r y'^p dx < \infty.$$

Moreover,

$$\int_{-x_1}^{x_1} su^{p-1}dx = 0$$

since su^{p-1} is an odd function. Proceeding from (2.1), as above, we have

$$\int_{-x_1}^{x_1} su^p dx > \int_{-x_2}^{x_2} sy^p dx = \int_{-x_2}^{x_2} ry'^p dx - ryy'^{p-1}\Big|_{-x_2}^{x_2}$$
$$> (1-\delta)\int_{-x_1}^{x_1} ru'^p dx,$$

provided

$$\delta \int_{-x_2}^{x_2} ry'^p dx > r\, y\, y'^{p-1}\Big|_{-x_2}^{x_2}.$$

By (2.20) this inequality can be satisfied for any $\delta > 0$ by taking x_2 sufficiently close to x_1. Hence (2.7) is sharp in this case.

Theorem 2.1, and its proof, remains valid if a or b are infinite, provided the order conditions are modified by replacing $(x - a)$ or $(b - x)$ by $|x|$. The same remark also applies to the preceding discussion of sharpness.

In our first three examples, $y(x)$ is admissible; in each case the second of conditions (2.3) holds, and we may use $k_1 = k_2 = \bar{x} = 0$ in (2.4), (2.5).

If

$$1 \leqslant k \leqslant n, \qquad \text{and} \qquad \int_{-\infty}^{\infty} x^{2(n-k)}(1 + x^{2n})^{-2k}u^{2k-1}dx = 0,$$

then

$$(2n+1)(2k-1)\int_{-\infty}^{\infty} \frac{x^{2(n-k)}u^{2k}dx}{(1+x^{2n})^{2k}} < \int_{-\infty}^{\infty} u'^{2k}dx \tag{2.21}$$

unless $u \equiv cx\ (1 + x^{2n})^{-1/(2n)}$.

If

$$k \geqslant 2, \qquad \text{and} \qquad \int_{-\infty}^{\infty} x^{2k/(2k-1)}(1 + x^{2k})^{-2k}u^{2k-1}dx = 0,$$

then

$$(2k-1)\left[1 - \frac{2k}{(2k-1)^2}\right]^{2k-1}\left[2k + 1 - \frac{2k}{(2k-1)^2}\right] \int_{-\infty}^{\infty} \frac{x^{2k/(2k-1)}u^{2k}dx}{(1+x^{2k})^{2k}} < \int_{-\infty}^{\infty} x^{2k/(2k-1)}u'^{2k}dx \tag{2.22}$$

unless $u \equiv cx^{\alpha}\ (1 + x^{2k})^{-\alpha/2k}$, where $\alpha = 1 - 2k/(2k-1)^2$.

If

$$k \geqslant 1, \qquad \text{and} \qquad \int_{-\infty}^{\infty} x^{2k/(2k-1)}(1 + x^{2k+2})^{-2k}u^{2k-1}dx = 0,$$

then

$$(2.23)\quad (2k-1)\left[1+\frac{2(k-1)}{(2k-1)^2}\right]^{2k-1}\left[2k+3+\frac{2(k-1)}{(2k-1)^2}\right]$$
$$\int_{-\infty}^{\infty}\frac{x^{2k/(2k-1)}u^{2k}dx}{(1+x^{2k+2})^{2k}} < \int_{-\infty}^{\infty} x^{-2(k-1)/2k-1}u'^{2k}dx$$

unless $u \equiv cx^{\alpha}\ (1+x^{2k+2})^{-\alpha/(2k+2)}$, where $\alpha = 1+2(k-1)/(2k-1)^2$. In (2.22), y' has a discontinuity at $x = 0$, as does $r(x)$ in (2.23).

In the next two examples, $y(x)$ is again admissible except in the extreme cases ($m = n = k$, and $n = k$, respectively) when hypotheses (2.20) are satisfied. Again the second of conditions (2.3) holds, and we take $k_1 = k_2 = \bar{x} = 0$ in (2.4), (2.5).

If

$$m > k,\ n > k, \text{ and } \int_{-1}^{1} x^{2(n-k)}\ (1-x^{2n})^{2(m-k)}\ u^{2k-1}dx = 0$$

then

$$(2.24)\quad [4mn-(2n+1)(2k-1)]\int_{-1}^{1} x^{2(n-k)}(1-x^{2n})^{2(m-k)}u^{2k}dx$$
$$< \int_{-1}^{1}(1-x^{2n})^{2m}u'^{2k}dx$$

unless $u \equiv cx\ (1-x^{2n})^{-1/(2n)}$, where $c = 0$ if $m = n = k$.

If

$$k > n > 1, \quad \text{and} \quad \int_{-1}^{1} u^{2k-1}dx = 0,$$

then

$$(2.25)\quad \left[1-\frac{2(k-n)}{2k-1}\right]^{2k-1}\int_{-1}^{1} u^{2k}dx < \int_{-1}^{1} x^{2(k-n)}(1-x^{2n})^{2k}u'^{2k}dx$$

unless $u \equiv cx^{\alpha}\ (1-x^{2n})^{-\alpha/(2n)}$, where $\alpha = 1-2(k-n)/(2k-1)$, and $c = 0$ if $n = k$.

An examination of the proof of Theorem 2.1 shows that the last two of conditions (2.2) as well as hypotheses (2.4), (2.5) were used to prove (2.15) for all admissible $u(x)$. Clearly *if the admissible $u(x)$ are also restricted to be bounded on (a, b), then these hypotheses may be replaced by the weaker hypothesis*

$$(2.26)\quad rh = r\left(\frac{y'}{y}\right)^{p-1} = o(1) \quad \text{as} \quad x \to a, \quad \text{or} \quad x \to b.$$

Moreover, if

$$\int ry'^{p}dx < \infty,$$

and y is bounded on (a, b), then (2.19) still holds, and $y(x)$ is admissible. We conclude this section with an example to which (2.26) applies, although Theorem 2.1 does not. If $n \geqslant k \geqslant 1$, $u(x)$ is bounded on $(-\infty, \infty)$, and

$$\int_{-\infty}^{\infty} x^{2(n-k)}(1 + x^{2n})^{-1} u^{2k-1} dx = 0,$$

then

$$(2.27) \qquad (2k - 1) \int_{-\infty}^{\infty} \frac{x^{2(n-k)} u^{2k} dx}{1 + x^{2n}} < \int_{-\infty}^{\infty} (1 + x^{2n})^{2k-1} u'^{2k} dx$$

unless $u \equiv cx(1 + x^{2n})^{-1/(1n)}$. (In this example, since $rh = x^{1-2k}$, we may even allow $u = O(|x|^{(2k-1/2k)})$ as $|x| \to \infty$.)

3. The special Wirtinger extension. We begin by defining a certain hyperelliptic function, $y(x)$, which satisfies a differential equation of the form (2.1) with $r(x)$, $s(x)$ constant (cf. (**3**, p. 182)). For $0 \leqslant x \leqslant \pi/2$, $y(x)$ is the unique solution of the equation

$$x = k \sin \frac{\pi}{2k} \int_0^y (1 - t^{2k})^{-1/(2k)} dt, \qquad 0 \leqslant y \leqslant 1.$$

We then define $y(x) \equiv y(\pi - x)$ for $\frac{1}{2}\pi \leqslant x \leqslant \pi$, and $y(x) \equiv -y(-x)$, $-\pi \leqslant x \leqslant 0$; in general, $y(x)$ is continued periodically (period 2π) for all x, and its graph is a curve similar in form to $y = \sin x$. One easily verifies that $y(x)$ is a solution of the differential equation

$$(3.1) \qquad \frac{d}{dx}\left\{\frac{1}{2k - 1}\left(k \sin \frac{\pi}{2k}\right)^{2k} y'^{2k-1}\right\} + y^{2k-1} = 0,$$

as is $y(x + \alpha)$ for any α.

THEOREM 3.1. *If $u(x)$ is an integral on $[-\pi, \pi]$, $u(-\pi) = u(\pi)$, and*

$$\int_{-\pi}^{\pi} u^{2k-1} dx = 0,$$

then

$$(3.2) \qquad \int_{-\pi}^{\pi} u^{2k} dx \leqslant \frac{1}{2k - 1}\left(k \sin \frac{\pi}{2k}\right)^{2k} \int_{-\pi}^{\pi} u'^{2k} dx,$$

and equality holds only if $u \equiv cy(x + \alpha)$, where $y(x)$ is the function defined above.

Proof. Here we cannot define $h = (y'/y)^{p-1}$ and proceed as in Theorem 2.1, because (2.15) is not valid. Instead, we shall use a different h in this case. First, the function $g(x) = u(\pi - x) - u(-x)$ is continuous for $0 \leqslant x \leqslant \pi$ with $g(\pi) = -g(0)$. Hence, there exists $\alpha(-\pi \leqslant -\alpha < 0)$, such that $u(\pi - \alpha) = u(-\alpha)$. Define $Y(x) \equiv y(x + \alpha)$, and set

$$h(x) = \{Y'(x)/Y(x)\}^{2k-1}.$$

Then $h(x) = O[(x+\alpha)^{1-2k}]$ near $x = -\alpha$, and similarly for x near $\pi - \alpha$. We now proceed as in the proof of Theorem 2.1, but taking $\bar{u} = u(-\alpha) = u(\pi - \alpha)$, and integrating the non-negative expression

$$r\{u' + (2k-1)\, h^{2k/(2k-1)}\, (u-\bar{u})^{2k} - 2k\, hu'\, (u-\bar{u})^{2k-1}\}$$

over the *three* intervals $[-\pi, -\alpha-\epsilon]$, $[-\alpha+\epsilon, \pi-\alpha-\epsilon]$, $[\pi-\alpha+\epsilon, \pi]$. The *two* results corresponding to (2.12) are clearly valid since the hypotheses corresponding to those ensuring (2.12) are also satisfied here. Thus we obtain (2.13), and the result corresponding to (2.14), namely,

$$\int_{-\pi}^{\pi} u^{2k} dx \leqslant C \int_{-\pi}^{\pi} u'^{2k} dx - C\left[\frac{y'(x+\alpha)}{y(x+\alpha)}\right]^{2k-1} [u(x) - u(-\alpha)]^{2k} \Bigg|_{-\pi}^{\pi}$$

where $C = r(x) = 1/(2k-1)\ (k \sin \pi/(2k))^{2k}$. Since $u(-\pi) = u(\pi)$, and y is periodic, the last term above vanishes and (3.2) is proved.

As for equality in (3.2), it holds only if $u(-\alpha) = 0$, and

$$u(x) = \begin{cases} c_1\, y(x+\alpha), & -\pi \leqslant x \leqslant -\alpha, \\ c_2\, y(x+\alpha), & -\alpha \leqslant x \leqslant \pi - \alpha, \\ c_3\, y(x+\alpha), & \pi - \alpha \leqslant x \leqslant \pi. \end{cases}$$

Since $u(-\pi) = u(\pi)$, we must have $c_3 = c_1$ (for $-\pi < -\alpha < 0$, as we are assuming). But then it follows from

$$\int_{-\pi}^{\pi} u^{2k-1} dx = 0, \qquad \text{and} \qquad \int_{-\pi}^{\pi} y^{2k-1}(x+\alpha)dx = \int_{-\pi}^{\pi} y^{2k-1}(x)dx = 0$$

that $c_2 = c_1$ also.

If $\alpha = \pi$, then $u(-\pi) = u(0) = u(\pi)$, and the original proof of Theorem 2.1 goes through with $h = (y'/y)^{2k-1}$, and $\bar{u} = u(0)$, using *two* intervals $[-\pi + \delta_1, -\epsilon]$ and $[\epsilon, \pi - \delta_2]$.

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SECTION III

Certain Properties of the Generalized Integral of a Finite Riemann Derivative*

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1. Introduction. The operator H_n was defined in a previous paper **(6)** as follows:

DEFINITION 1.1. Let $F(x)$ be any single valued function defined over a given domain. Then

$$H_n(F: x_1, \ldots, x_{n+1}) = w_{n+1}(x_{n+1}) \sum_{j=1}^{n+1} \frac{F(x_j)}{w'_{n+2}(x_j)} \qquad (n = 1, 2, \ldots,) \tag{1.1}$$

where

$$w_{n+1}(x) = \prod_{i=1}^{n} (x - x_i)$$

and the "prime" denotes ordinary differentiation.

Now, let $f(x)$ be a measurable function defined in the interval (a, b), and consider the expression

$$\Delta_n(x, 2h; f) = H_n(f: x - nh, \ldots, x + 2jh - nh, \ldots, x + nh) \qquad (j = 0, 1, \ldots, n).$$

If the limit of $(2h)^{-n}\Delta_n(x, 2h; f)$ exists and is finite at the point x, as $h \to 0$, it is called the nth generalized Riemann derivative of $f(x)$ at the point x, $D^n f(x)$.

By means of H_n we define a generalized integral as follows:

DEFINITION 1.2. Let $f(x)$ be defined on $[a, b]$ and be such that there exists a continuous function $F(x)$ for which $D^n F(x) = f(x)$, $x \in (a, b)$. If $F(x)$ is such that the *Dini-Lebesgue* theorem is satisfied, namely, that

$$\inf_{a<x<b} D^n F(x) \leqslant \frac{n! H_n(F: p_1, \ldots, p_{n+1})}{(p_{n+1} - p_1) \ldots (p_{n+1} - p_n)} \leqslant \sup_{a<x<b} D^n F(x) \tag{1.2}$$

for every $n + 1$ distinct points of $[a, b]$, $p_1, \ldots, p_{n+1}$, then $f(x)$ is called *n-integrable* on $[a, b]$ and the *n-integral* $I_n(f: x_1, \ldots, x_n, x)$ of $f(x)$ on $[a, b]$ is defined by

$$I_n(f: x_1, \ldots, x_n, x) = H_n(F: x_1, \ldots, x_n, x) \tag{1.3}$$

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where $x_1, \ldots, x_n$ are n arbitrary distinct points of $[a, b]$ and $a \leqslant x \leqslant b$.

The following properties of H_n are direct consequences of Definition 1.1.

(i) $H_n(F: x_1, \ldots, x_n, x) - H_n(F: x_1', \ldots, x_n', x)$ reduces to a polynomial in x of degree at most $n - 1$.

(ii) $H_n(F: x_1, \ldots, x_{n+1})$ vanishes when $x_{n+1} = x_j$ $(j = 1, 2, \ldots, n;$ $n \geqslant 1)$.

(iii) $$H_n\left(\sum_{i=1}^{m} a_i F_i : x_1, \ldots, x_{n+1}\right) = \sum_{i=1}^{m} a_i H_n(F_i: x_1, \ldots, x_{n+1})$$

($\{a_i\}$ constants).

(iv) $H_n(F: x_1, \ldots, x_n, x_{n+1})$ remains invariant under all permutations of the points $x_1, \ldots, x_n$.

In this paper we obtain necessary and sufficient conditions for the vanishing of the operator H_n and we establish an additivity law which is satisfied by H_3. The method can be extended to the operator H_n but the operations become very involved. The additivity law for H_2 was developed by Denjoy (**1**, pp. 278–279). A related operation for H_2 was obtained by James (**5**, Theorem 14).

The generalized integrals are of interest because of their applications to the problem of the determination of the coefficients of trigonometric series which converge in a general sense. In particular, if the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

converges everywhere to the function $f(x)$ then it is known that

$$a_k = \frac{1}{2\pi^2} I_2(f(x) \cos kx: -2\pi, 0, 2\pi);\ b_k = \frac{1}{2\pi^2} I_2(f(x) \sin kx: -2\pi, 0, 2\pi).$$

This result follows from Denjoy's work (**1**). It also follows from results of James (**2**, pp. 305–306) who developed an integral in terms of major and minor functions similar to the Perron integral. In later papers (**3**) and (**4**, §8) James extended his results to orders higher than 2 by relating his integrals to derivatives similar to de la Vallée Poussin derivatives.

The applications of the integral I_n to trigonometric series will be considered in another paper.

2. Conditions for $H_n = 0$.

THEOREM 2.1. *The necessary and sufficient condition for the vanishing of*

$$H_n(F: p_1, \ldots, p_{n+1})$$

for every $n+1$ points $p_1, \ldots, p_{n+1}$ of $[a, b]$ where $p_1, \ldots, p_n$ are distinct points, is that $F(x)$ be a polynomial of degree at most $n-1$ on $[a, b]$.

The condition is sufficient. According to (1.1), $H_n(F: p_1, \ldots, p_n, x)$ is a polynomial in x of degree at most $n-1$ which has n roots $p_1, \ldots, p_n$. Hence $H_n(F: p_1, \ldots, p_n, x)$ is identically zero and consequently

$$H_n(F: p_1, \ldots, p_n, p_{n+1}) = 0.$$

The condition is necessary. Here we can prove a stronger result by considering the points $p_1, \ldots, p_{n+1}$ to be in arithmetic progression on the interval $[a, b]$. $F(x)$ is supposed to be continuous on $[a, b]$. To prove this result we consider the equation

$$H_n(F: p_1, \ldots, p_n, x) = 0 \qquad (a \leqslant x \leqslant b)$$

where $p_1, p_2, \ldots, p_n$ are in arithmetic progression. According to (1.1) we have

$$F(x) - P_{n-1}(x) = 0 \tag{2.1}$$

where $P_{n-1}(x)$ is a polynomial in x of degree at most $n-1$ whose coefficients are functions of the n points $p_1, \ldots, p_n$. Equation (2.1) admits as solutions the $n+1$ points of the arithmetic progression $p_1, \ldots, p_n, p_{n+1}$. It follows that for every $n+1$ points of $[a, b]$ in arithmetic progression there exists a corresponding polynomial of degree at most $n-1$ whose graph has in common with that of $F(x)$ the points $[p_1 F(p_1)], \ldots, [p_{n+1}, F(p_{n+1})]$.

We subdivide the interval $[a, b]$ into K equal subintervals ($K > n$) by means of the $K+1$ points $a = M_1, M_2, \ldots, M_K, M_{K+1} = b$; then we construct the sequence of polynomials $P_{n-1}{}^i(x)$ so that the graph of $P_{n-1}{}^i(x)$ has in common with that of $F(x)$ the $n+1$ points

$$[M_i, F(M_i)], \ldots, [M_{i+n}, F(M_{i+n})] \ (i = 1, 2, \ldots, K-n+1).$$

The polynomials $P_{n-1}{}^i(x)$ coincide with a unique polynomial $P_{n-1}(x)$ because the graphs of any two consecutive polynomials of this sequence have n points in common. The graph of the polynomial $P_{n-1}(x)$ has in common with the graph of $F(x)$ the $K+1$ points $[M_1, F(M_1)], \ldots, [M_{K+1}, F(M_{K+1})]$.

We proceed to a new subdivision of $[a, b]$ by means of the previous $K+1$ points together with the midpoints of the subintervals $[M_1, M_2], \ldots, [M_K, M_{K+1}]$. This new subdivision is formed by $2K$ subintervals of equal length whose extremities are the $2K+1$ points $a = N_1, N_2, \ldots, N_{2K+1} = b$; then again, we construct the sequence of polynomials $Q_{n-1}{}^i(x)$ so that the graph of $Q_{n-1}{}^i(x)$ has in common with that of $F(x)$ the $n+1$ points $[N_i, F(N_i)], \ldots, [N_{i+n}, F(N_{i+n})]$ $(i = 1, 2, \ldots, 2K-n+1)$ and we prove, in exactly the same way as previously, that the polynomials $Q_{n-1}{}^i(x)$ coincide with a unique polynomial $Q_{n-1}(x)$ whose graph has $2K+1$ points in common with that of $F(x)$, namely, the points

$$[N_1, F(N_1)], \ldots, [N_{2K+1}, F(N_{2k+1})].$$

Evidently, $P_{n-1}(x)$ is identical with $Q_{n-1}(x)$ since their graphs have $K + 1$ points in common and $K + 1 > n - 1$.

Proceeding in this way, we can determine a polynomial of degree at most $n - 1$ whose graph has in common with that of $F(x)$ all the points of an everywhere dense set on $[a, b]$, and since $F(x)$ is continuous on $[a, b]$ it follows that $F(x)$ reduces to a polynomial of degree at most $n - 1$ on $[a, b]$.

It is difficult to conceive of a process of integration which would produce a non-zero integral for a function that is identically zero. This is settled in the following

COROLLARY 2.1. *If $f(x)$ is identically zero on $[a, b]$ then $f(x)$ is n-integrable on $[a, b]$ $(n = 1, 2, \ldots,)$, and*

$$I_n(f: x_1, \ldots, x_n, x) = 0$$

where $x_1, \ldots, x_n$ are n arbitrary distinct points of $[a, b]$ and $a \leqslant x \leqslant b$. The converse is true also.

Proof. Consider the class of continuous functions which satisfy Definition 1.2, with $f(x) = 0$ for all $x \in [a, b]$. This class is not empty since it contains the arbitrary polynomial in x of degree at most $n - 1$. By relation (1.2), it follows that if $F(x)$ belongs to this class of functions then

$$H_n(F: x_1, \ldots, x_n, x) = 0.$$

Hence, by relation (1.3), we obtain

$$I_n(f: x_1, \ldots, x_n, x) = 0.$$

Proof of the converse. Let $f(x)$ be n-integrable on $[a, b]$ and such that $f(x)$ is not identically zero. Moreover,

$$I_n(f: x_1, \ldots, x_n, x) = 0$$

for every n distinct points on $[a, b]$, $x_1, \ldots, x_n$, and $a \leqslant x \leqslant b$. Then there exists a function $F(x)$ satisfying the requirements of Definition 1.2 and such that

$$I_n(f: x_1, \ldots, x_n, x) = H_n(F: x_1, \ldots, x_n, x) = 0.$$

By Theorem 2.1, it follows that $F(x)$ reduces to a polynomial of degree at most $n - 1$ and hence $D^n F(x) \equiv f(x) \equiv 0$, which contradicts the hypothesis on $f(x)$.

3. An additivity law for the operator H_3. We say that we can calculate the operator H_3 for the function $F(x)$ over the closed or open interval **i** if we can find the value of the expression $H_3(F: p_1, p_2, p_3, p_4)$ for every four points of the interval **i**, where p_1, p_2, p_3 are distinct points.

Consider now the closed interval $[a, b]$ and let p be a fixed arbitrary point such that $a < p < b$. Let $F(x)$ be a function defined and continuous

on $[a, b]$ and possessing a finite third generalized Riemann derivative at each point of (a, b).

PROBLEM 3.1. *Calculate the operator H_3 for the function $F(x)$ over the interval $[a, b]$, given that it is known how to calculate H_3 over the intervals $[a, p + h]$ and $[p - h, b]$ separately, where h is an arbitrary positive variable number destined to tend to zero.*

In order to solve this problem we consider the identity

$$(3.1) \quad H_3(F: x_1, x_2, x_3, x_4) = H_3(F: p + h, x_2, x_3, x_4) - \frac{(x_4 - x_2)(x_4 - x_3)}{(x_1 - x_2)(x_1 - x_3)} H_3(F: p + h, x_2, x_3, x_1)$$

where x_1, x_2, x_3, x_4 are four arbitrary points of $[a, b]$, x_1, x_2, x_3 are distinct points, and

$$h < \min\left\{\frac{|p - x_1|}{3}, \frac{|p - x_2|}{3}, \frac{|p - x_3|}{3}\right\}.$$

We distinguish two cases: (i) one point belongs to the half-open interval $[a, p)$ and the remaining three points belong to $(p, b]$; (ii) two points belong to $[a, p)$ and the remaining two points belong to $(p, b]$.

Case (i). Let $x_1 \in [a, p)$ and $\{x_2, x_3, x_4\} \in (p, b]$. Then the first term in the right side of (3.1) is known by the hypothesis of Problem 3.1. To calculate the second term in the right side of (3.1) we use the identity

$$\begin{aligned} (3.2) \quad H_3(F: p + h, x_2, x_3, x_1) &= H_3(F: p - 3h, p - h, p + h, x_1) \\ &- [(x_1 - p)^2 - h^2](8h^2)^{-1} H_3(F: p - 3h, p - h, p + h, p + 3h) \\ &- \frac{(x_1 - p - h)(x_1 - x_3)}{(x_2 - p - h)(x_2 - x_3)} H_3(F: p - h, p + h, p + 3h, x_2) \\ &- \frac{(x_1 - p - h)(x_1 - x_2)}{(x_3 - p - h)(x_3 - x_2)} H_3(F: p - h, p + h, p + 3h, x_3). \end{aligned}$$

The last three terms in the right side of (3.2) are known by the hypothesis of Problem 3.1. The first term in the right side of (3.2) is unknown, but it tends to zero with h. Indeed, it is known that

$$\lim_{h \to 0} (8h^2)^{-1} H_3(F: p - 3h, p - h, p + h, p + 3h) = 0 \qquad (a < p < b)$$

when $F(x)$ possesses a finite third generalized derivative in (a, b). Consequently, if we rewrite (3.2) in the shorter form

$$H_3(F: p + h, x_2, x_3, x_1) = g(h) + g_2(h) + g_3(h) + g_1(h) = g(h) + G(h)$$

we obtain

$$(3.3) \qquad \lim_{h \to 0} H_3(F: p + h, x_2, x_3, x_1) = \lim_{h \to 0} G(h).$$

Hence, by (3.1)

$$H_3(F: x_1, x_2, x_3, x_4) = \lim_{h\to 0} H_3(F: p+h, x_2, x_3, x_4) - \frac{(x_4 - x_2)(x_4 - x_3)}{(x_1 - x_2)(x_1 - x_3)} \lim_{h\to 0} G(h).$$

Since $F(x)$ is continuous on $[a, b]$ we obtain

$$(3.4)\quad H_3(F: x_1, x_2, x_3, x_4) = H_3(F: p, x_2, x_3, x_4) - \frac{(x_4 - x_2)(x_4 - x_3)}{(x_1 - x_2)(x_1 - x_3)} \lim_{h\to 0} G(h).$$

Case (ii). Let $\{x_1, x_4\} \in [a, p)$ and $\{x_2, x_3\} \in (p, b]$. We use again relation (3.1). We know already the limit, as $h \to 0$, of the second term in the right side of (3.1). In order to calculate the first term in the right side of (3.1) we use (3.2) after having substituted in it x_4 for x_1. Thus, we arrive at a relation that corresponds to (3.4) of case (i).

In case one of the four points coincides with p, say the point x_4, we can obtain the operator H_3 from relation (3.3) where we substitute x_4 for p. We have

$$H_3(F: x_4, x_2, x_3, x_1) = \lim_{h\to 0} G(h),$$

and using the identity

$$H_3(F: x_1, x_2, x_3, x_4) = -\frac{(x_4 - x_2)(x_4 - x_3)}{(x_1 - x_2)(x_1 - x_3)} H_3(F: x_4, x_2, x_3, x_1)$$

we obtain finally

$$H_3(F: x_1, x_2, x_3, x_4) = -\frac{(x_4 - x_2)(x_4 - x_3)}{(x_1 - x_2)(x_1 - x_3)} \lim_{h\to 0} G(h).$$

Relation (3.4) and the corresponding relations of the other cases establish an *additivity law* for the operator H_3. This law can be expressed immediately in terms of the symbol I_3 by means of relation (1.3). We obtain thus an additivity law for the corresponding generalized integral, provided that $D^3F(x)$ is *3-integrable* on the intervals over which this integral is calculated.

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